

Question 1) Define conjugate elements.

Prove that the relation of conjugacy is an equivalence relation on G .

Answer Conjugate elements (Defn.):— If a and b be two elements of a group G . Then b is said to be conjugate to a if there exists an element $x \in G$ such that

$$b = x^{-1}ax.$$

If $b = x^{-1}ax$, then b is also called the transform of a by x .
If b is conjugate to a then symbolically we shall write $b \sim a$ and this relation in G is called the relation of conjugacy.

Thus $b \sim a$ iff $b = x^{-1}ax, \forall x \in G$.

Proof. We have to prove the theorem that the relation of conjugacy is an equivalence relation on G .

In order to prove this, we proceed as follows:

Reflexivity: If a is any element of G , then we have

$$a = e^{-1}ae \Rightarrow a \sim a.$$

Thus $a \sim a, \forall a \in G$. Thus the relation is reflexive.

Symmetry: we have,

$$a \sim b \Rightarrow a = x^{-1}bx \text{ for some } x \in G.$$

$$\Rightarrow xax^{-1} = x(x^{-1}bx)x^{-1} \Rightarrow xax^{-1} = b \Rightarrow b = (x^{-1})^{-1}ax^{-1}$$

Therefore, we get the relation is symmetric where $x^{-1} \in G \Rightarrow b \sim a$.

Transitivity Let $a \sim b, b \sim c$,

$$\text{Then } a = x^{-1}bx, b = y^{-1}cy \quad \forall x, y \in G.$$

From this we have

$$a = x^{-1}(y^{-1}cy)x \quad [\because b = y^{-1}cy].$$

$$= (yx)^{-1}c(yx), \text{ where } yx \in G.$$

$\therefore a \sim c$ and thus we find that the relation is transitive.

Hence we conclude that the relation of conjugacy in a group G is an equivalence relation. Therefore,

it will partition G into disjoint equivalence classes called classes of conjugate elements.

Note: Classes of conjugate elements will be such that

(i) any two elements of the same class are conjugate.

(ii) no two elements of different classes are conjugate.

The collection of all elements conjugate to an element $a \in G$ will be symbolically denoted by $C(a)$ or by \bar{a} . Thus $C(a) = \{x \in G; x^{-1}ax\}$.

$C(a)$ will be called the conjugate class of a in G . we have $(y^{-1}ay) \sim a, \forall y \in G$. Also if $b \sim a$ then b must be equal to $y^{-1}ay, \forall y \in G$. Therefore

$C(a) = \{y^{-1}ay; y \in G\}$. If G is a finite group, then the number of distinct elements in $C(a)$ will be denoted by C_a .

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